VII. *Direct and expeditious methods of calculating the excentric from the mean anomaly of a planet.* *By the Reverend* Abram Robertson, *D. D. F. R. S. Savilian Professor of Astronomy in the University of Oxford, and Radcliffian Observer. Communicated by the Right Hon. Sir* Joseph Banks, *Bart. G. C. B. P. R. S.*

Read February 15, 1816.

SINCE the publication of KEPLER's discoveries in astronomy, the attention of men of science has frequently been directed to the problem distinguished by his name, and their exertions have frequently been employed to overcome the acknowledged difficulty of its solution. A statement of the various degrees of success, with which these endeavours have been made, is foreign to the present design. An account of this kind is now also needless, as Dr. BRINKLEY's examination of such attempts, published in the ninth volume of the Transactions of the Royal Irish Academy, affords a satisfactory review of most of the proceedings on this subject, previous to the year 1802.

After the following methods had occurred to my consideration, and I had fully proved their utility by actual application to examples, I was anxious to ascertain whether any author had anticipated me in the manner in which the investigations are conducted. With this view I examined such solutions as are referred to in Dr. BRINKLEY's very able Memoir, all those mentioned by MONTUCLA,* of which I could procure a sight,

* Histoire des Mathematiques, Tom. II. p. 343, &c. I have searched, without success, for Lorgna's and Trembley's publications.

and some others which had occurred to me in the course of my reading. The result of this search is a belief that no one before has aimed at a direct solution through the same small angle, and, by means of an equation in which this angle and its powers are the only unknown quantities, obtained a quickly converging series for its value in known terms. The small angle being found, with due precision, the eccentric anomaly readily becomes known.

M. DELAMBRE, in his Astronomy, published at Paris in 1814, in three quarto volumes, calculates the excentric anomaly by a method founded on those of CASSINI,* LA CAILLE,† SIMPSON,‡ and CAGNOLI.§ This eminent astronomer says of it, "Ce procédé, le plus directe que je connaises, est aussi le plus précis; il n'est qu'approximatif, mais il est toujours exact au-delà des dixièmes de seconde pour toutes les planètes de notre système." That the reader may readily judge how far the third method, which I now propose, deserves attention, I have annexed to my investigation the calculation by it of M. DELAMBRE's two examples, and also one relating to the comet of 1682 and 1759.

Each of the following methods of solution is to be considered as direct, although it proceeds through the medium of what is commonly called CASSINI's approximation. This approximation, as here used, can only be considered as the first certain step in the computation. No hypothesis is introduced

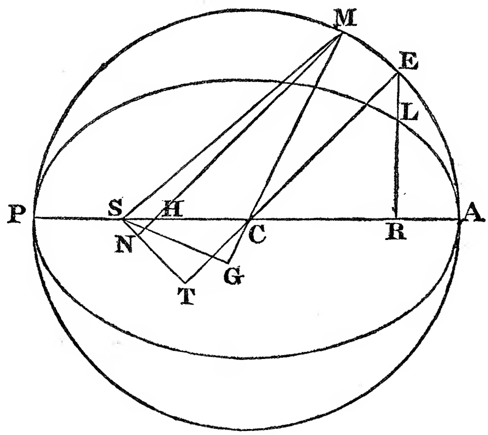
* Mémoires de l'Academie de l'année 1719.

† Leçons Elémentaires d'Astronomie, Paris, 1761.

‡ Essays on several curious and useful subjects, London, 1740.

§ Trigonométrie, Paris, 1786.

into the proceeding, and therefore no correction by trial and error is requisite.



Let ALP be the orbit of a planet, C the centre of the ellipse, S that focus in which the sun is placed, and AMP a circle described on the greater axis AP as a diameter. Let L be the true place of the planet, and AM the corresponding mean anomaly. Through L let the straight line ER be drawn, perpendicular to AP, and let it meet AP in R and the circle in E. Let EC, CM, SM be drawn, and let ST be perpendicular to EC, SG to CM and MN to ST. Then MN is parallel to ET, and NT is equal to the sine of the arc EM. It is easily proved, as in almost every writer on the subject, that ST is equal to the arc EM.

In this Problem it is supposed that AC, CS, AM are given, and it is required to find AE the excentric anomaly, for AE being found, the true anomaly ASL is easily obtained.

In each of the three following methods the angles CMS, CSM are used, and their difference is found by this proportion.

$CM + CS : CM - CS :: \tan. \frac{1}{2} ACM : \tan. \frac{1}{2} (CSM - CMS)$. Hence the angles become known by their sum and difference. As the angle SMN is very small, and consequently the angle MSC nearly equal to ECA in the orbits of almost all the planets, this way of finding the angle CSM is usually called CASSINI's approximation to the excentric anomaly ACE.

FIRST METHOD.

Having found the angle CSM, sin. CSM : CM :: sin. SCM : SM, which therefore becomes known. Let z equal the angle SMN, s equal the series expressing its sine, and c equal the series expressing its cosine. Put a equal the sine of CMS, and b equal its cosine. Then, radius being 1, $ac - bs =$ sin. CMN $=$ sin. ECM $=$ sin. $(CMS - z)$. Also, $1 : SM :: s : SN = SM \times s$, and $ac - bs + SM \times s = TN + SN = EM = CMS - z$, and therefore $CMS = z + ac - bs + SM \times s = z + ac + \overline{SM - b} \times s$.

Let $d = SM - b$, and then $CMS = z + ac + ds = z + a\left(1 - \frac{z^2}{2} + \frac{z^4}{2.3.4} - \frac{z^6}{2.3.4.5.6} + \frac{z^8}{2.3.4.5.6.7.8} - \&c.\right) + d\left(z - \frac{z^3}{2.3} + \frac{z^5}{2.3.4.5} - \&c.\right) = a + z + dz - \frac{az^2}{2} - \frac{dz^3}{2.3} + \frac{az^4}{2.3.4} + \frac{dz^5}{2.3.4.5} - \&c.$

Let $e = CMS - a$, and putting A, B, C, &c. for the coefficients $e = Az - Bz^2 - Cz^3 + Dz^4 + Ez^5 - Fz^6 - Gz^7 + \&c.$

By reversing this series we find $z = \frac{1}{1+d} e + \frac{a}{2(1+d)^3} e^2 + \frac{d}{6(1+d)^4} e^3 + \frac{a^2}{2(1+d)^5} e^3 - \frac{a}{24(1+d)^5} e^4 + \frac{5ad}{12(1+d)^6} e^4 + \frac{5a^3}{8(1+d)^7} e^4 + \&c.$

This equation is in parts of the radius, and in order to have it in degrees we use this proportion, $1 : 57^\circ.2957795 :: z : 57^\circ.2957795 z = R^\circ z$, putting R° for $57^\circ.2957795$.

Hence $R^{\circ}z = \frac{R^{\circ}}{1+d}e + \frac{R^{\circ}a}{2(1+d)^3}e^2 + \frac{R^{\circ}d}{6(1+d)^4}e^3 + \frac{R^{\circ}a^2}{2(1+d)^5}e^3 +$ &c.

SECOND METHOD.

The substitutions for the sines and cosines of the angles CMS, SMN being as in the preceding method, let SG be perpendicular to MC, and then radius being 1, 1 : CS : : sin. ACM : CS $\times$ sin. ACM $=$ SG. But sin. CMS : SG : : sin. SMN : SN, that is, a : CS $\times$ sin. ACM : : s : $\frac{\text{CS} \times \text{sin. ACM}}{a}s =$ SN, and therefore $ac - bs + \frac{\text{CS} \times \text{sin. ACM}}{a}s =$ TN $+$ SN $=$ ST $=$ CMS $- z$. Let $d = \frac{\text{CS} \times \text{sin. ACM}}{a} - b$, and then $ac + ds =$ CMS $- z$, and CMS $= z + ac + ds$, as in the preceding method.

THIRD METHOD.

Let $z =$ the angle SMN, $s =$ the series expressing its sine, and $c =$ the series expressing its cosine, as before; but let a now denote the sine of CSM, and b its cosine, and let MN meet CS in H. Then $ac + bs$ is equal to the sine of the sum of the angles SMH, MSC, that is, $ac + bs =$ sin. MHA $=$ sin. ACE, the excentric anomaly. We have therefore CE : $ac + bs$: : CS : $\frac{\text{CS}(ac + bs)}{\text{CE}} =$ ST $=$ CMS $- z$.

Let $d = \frac{\text{CS} \times a}{\text{CE}}$, and $e = \frac{\text{CS} \times b}{\text{CE}}$, and then $dc + es =$ CMS $- z$, and CMS $= z + dc + es = z + d - \frac{dz^2}{2} + \frac{dz^4}{2.3.4} - \frac{dz^6}{2.3.4.5.6} +$ &c. $+ ez - \frac{ez^3}{2.3} + \frac{ez^5}{2.3.4.5} -$ &c. $= d + z + ez - \frac{dz^2}{2} - \frac{ez^3}{2.3} + \frac{dz^4}{2.3.4} + \frac{ez^5}{2.3.4.5} -$ &c. Let $f =$ CMS $- d$, and then $f = Az - Bz^2 - Cz^3 + Dz^4 + Ez^5 -$ &c. putting A, B, C, &c. for the coefficients. By reversing this series, or by putting d for a,

e for d, and f for e, in the series in the first method, we find

$$R^\circ z = \frac{R^\circ}{1+e}f + \frac{R^\circ d}{2(1+e)^3}f^2 + \frac{R^\circ e}{6(1+e)^4}f^3 + \frac{R^\circ d^2}{2(1+e)^5}f^3 - \frac{R^\circ d}{24(1+e)^5}f^4 + \frac{5R^\circ de}{12(1+e)^6}f^4 + \frac{5R^\circ d^3}{8(1+e)^7}f^4 + \&c.$$

I prefer this method to the first or second, and therefore I proceed to calculate by it.

EXAMPLE I.

Let us suppose with M. DELAMBRE* that the mean anomaly is 135°, and the excentricity of the orbit 0.25, the mean distance from the sun being 1.

Here CM+CS=1.25, CM−CS= .75, and $\frac{CM-CS}{CM+CS} = \frac{.75}{1.25}$, the log. of which is to be used for any given mean anomaly in the orbit.

$\frac{.75}{1.25}$ - -	Log. 9.7781513	CMS is found by this proportion, 206264″.8 : 1 : : CMS in seconds : its length in parts of the radius.
Log. tan. 67°..30′ -	10.3827757	
Log. tan. 55..22..49.84	10.1609270	
CSM = 122..52..49.84		

CMS = 12.. 7..10.16=43630″.16 - Log. 4.6397867
206264.8 - Log. 5.3144251

CMS = . 2115249 Log. 9.3253616

CS = 0.25	Log. 9.3979400	CS - -	Log. 9.3979400
a	Log. 9.9241783	b - -	Log. 9.7347108
d=.2099511	Log. 9.3221183	e=.1357222	Log. 9.1326508.
.2115249 = CMS		As CSM is obtuse, e is negative.	
.0015738=f,	Log. 7.1969495	1+e=.8642778	Log. 9.9366534.

* Vol. II. page 28.

The angle SMN is therefore calculated from the series in the following manner:

First term.				Second term.		
R° - - - -		Log.	1.7581226	R° - -	Log.	1.7581226
f - - - - -		Log.	7.1969495	d - -	Log.	9.3221183
			8.9550721	$f.^2$ - -	Log.	4.3938990
$1+e$ - - -		Log.	9.9366534			15.4741399
Numb. -	.1043322	Log.	9.0184187	2, Log. - -		0.3010300
For 2d term	.0000231					15.1731099
Sum -	0°.1043553 =		6′..15.″67	$(1+e)^3$ -	Log.	9.8099602
= SMH. CSM -	=		122°..52..49.84	Numb. .0000231	Log.	5.3631497
ACE = - -			122..59..5.51			

This differs from M. Delambre's conclusion only in the second place of the decimals.

Example II.

Supposing, with M. Delambre, that the mean anomaly in the same orbit is 96°, required the excentric anomaly.

We have as before	$\frac{.75}{1.25}$ - - - - - -	Log.	9.7781513
Log. tan.	48°.. - - - - - - - - -		10.0455626
Log. tan.	33 .40′..41″.51 - - - - - -		9.8237139
CSM =	81 .40 41″.51		
CMS =	14..19..18.49 = 51558.49	Log.	4.7123003
	206264.8	Log.	5.3144251
	CMS = .2499627	Log.	9.3978752

CS = 0.25	- -	Log.	9.3979400
a	- - -	Log.	9.9954030
d = .2473677	-	Log.	9.3933430
.2499627 = CMS			
.0025950 = f	-	Log.	7.4141374

First term of the series.

R°	- - -	Log.	1.7581226
f	- - -	Log.	7.4141374
			9.1722600
$(1 + e)$	- -	Log.	.0154366
Num.	.1434906	Log.	9.1568234
2d term	.0000429		
Sum	0°.1435335	=	8′..36″.72
= SMN. CSM	-	=	81°..40..41 .51
ACE	-	=	81 ..49..18 .23

CS = 0.25		Log.	9.3979400
b	- -	Log.	9.1605669
e = .0361832		Log.	8.5585069
$1+e$=1.0361832		Log.	.0154366

Second term of the series.

R°	- -	Log.	1.7581226
d	- -	Log.	9.3933430
f^2	- -	Log.	4.8282748
			15.9797404
2	- -	Log.	.3010300
			15.6787104
$(1 + e)^3$		Log.	.0463098
Num. .0000429		Log.	5.6324006

This also differs from M. DELAMBRE's conclusion only in the second place of decimals.

EXAMPLE III.

Let us suppose ALP to be the orbit of the comet which appeared in 1682, and reappeared in 1759, according to the prediction of Dr. HALLEY; that CE is equal to 18.07575, that that CS is equal to 17.49225, and that the mean anomaly is 179°..47′..32″.17, it is required to find the excentric anomaly.

Here $\frac{CM-CS}{CM+CS} = \frac{.5835}{35.568}$		- - -	Log.	8.2149815
Log. tan.	89°..53′..46″.08	- -		12.7415531
Lon. tan.	83 ..41 ..34 .5	- -		10.9565346
CSM =	173 ..35 ..20 58			
CMS =	6 ..12 ..11 58 = 22331.58		Log.	4.3489194
	206264.8		Log.	5.3144251
	CMS = .1082666		Log.	9.0344943

CS = 17.49225		Log.	1.2428457
a	- -	Log.	9.0478932
			10.2907389
CE = 18.07575		Log.	1.2570963
d = .1080544		Log.	9.0336426
.1082666 = CMS			
.0002122 = f,		Log.	6.3267454

First term of the series.

R°	- -	Log.	1.7581226
f	- - -	Log.	6.3267454
			8.0848680
$(1+e)$	- -	Log.	8.5835638
Num. .3171788		Log.	9.5013042

Second term of the series.

R°	- -	Log.	1.7581226
f^2	- -	Log.	2.6534908
d	- - -	Log.	9.0336426
			13.4452560
2	- -	Log.	.3010300
			13.1442260
$(1+e)^3$	-	Log.	5.7506914
Num. .0024748		Log.	7.3935346

.0024748
.3171788
.0000386

.3196922 sum of the positive
— .0000406

CS = 17.49225		Log.	1.2428457
b	- -	Log.	9.9972757
			11.2401214
CE = 18. 07575		Log.	1.2570963
e = . 9616678		Log.	9.9830251

As CSM is obtuse, e is negative.

$1+e$ = .0383322		Log.	8.5835638

Third term of the series.

R°	- -	Log.	1.7581226
e	- -	Log.	9.9830251
f^3	- -	Log.	8.9802362
			20.7213839
6	- -	Log.	.7781513
			19.9432326
$(1+e)^4$	-	Log.	4.3342552
Num. — .0000406		Log.	5.6089774

Fourth term of the series.

R°	- -	Log.	1.7581226
d^2	- -	Log.	8.0672852
f^3	- -	Log.	8.9802362
			18.8056440
2	- -	Log.	.3010300
			18.5046140
$(1+e)^5$	- -	Log.	2.9178190
Num. .0000386		Log.	5.5867950

SMN = 0°.3196516 = 19′..10″.75
CSM = 173..35..20 .58

173..54..31 .33 = ACE. With the design already

expressed, I adopted in this example the same data with Mr. IVORY: see Transactions of the Royal Society of Edinburgh, Vol. V. page 236.

The preceding method bears a nearer resemblance to that given by KEILL, in his Astronomy, than to any other of which I know. Adapting his manner of proceeding to the figure here used, he puts $y =$ EM, $e =$ sin. AM, f cos. AM, and $g =$ CS. The series expressing the sine of AE, is therefore equal to $e - fy - \frac{ey^2}{2} + \frac{fy^3}{2.3} + \frac{ey^4}{2.3.4}$ &c. But the radius, which is 1, is to the sine of AE as CS or g is to ST or EM, that is to y. Consequently $y = ge - gfy - \frac{gey^3}{2} + \frac{gfy^3}{2.3} + \frac{gey^4}{2.3.4}$ &c. and therefore $ge = y + gfy + \frac{gey^2}{2} - \frac{gfy^3}{2.3} - \frac{gey^4}{2.3.4}$ &c.

By reversing this he obtains a series, which, omitting the numbers in the coefficients, converges as the powers of $\frac{ge}{1 + gf}$ or $\frac{\text{CS} \times \text{sin. AM}}{1 + \text{CS} \times \text{cos. AM}}$. This degree of convergency to the value of y in the foregoing examples is as follows.

In the first example as the powers of .2147372,
In the second example as the powers of .2553020,
In the third example as the powers of .1086700.

In the third method which has been here investigated, the series converges to the value of SMH as the powers of $\frac{f}{1+e}$ or $\frac{\text{CMS} - \text{CS} \times \text{sin. CSM}}{1 + \text{CS} \times \text{cos. CSM}}$. This degree of convergency in the foregoing examples is as follows.

In the first example as the powers of .0018209,
In the second example as the powers of .0250438,
In the third example as the powers of .0055358.

The third method, by which the three foregoing examples

are calculated, appears to me the most simple and precise in theory, and the most expeditious in practice of any which I have seen. This I say with the greater degree of freedom, as I am so well aware of the similarity between its series and that of KEILL's, and so perfectly convinced of the advantages which it derives from CASSINI's approximation, that I consider it, with the exception of some deviations, as a combination of their methods.